# CAC with Nonlinearly-Constrained Feasibility Regions

Marco Cello, *Student Member, IEEE*, Giorgio Gnecco, Mario Marchese, *Senior Member, IEEE*, and Marcello Sanguineti

*Abstract*—Two criteria are proposed to characterize and improve suboptimal coordinate-convex (c.c.) policies in Call Admission Control (CAC) problems with nonlinearly-constrained feasibility regions. Then, a structural property of the optimal c.c. policies is derived. This is expressed in terms of constraints on the relative positions of successive corner points.

*Index Terms*—Call admission control, feasibility region, coordinate convex policies.

## I. INTRODUCTION

CALL Admission Control (CAC) is a tool of topical importance to guarantee a specific Quality of Service (QoS) over telecommunications networks for which it is possible to define the concept of call (also called connection). Phone calls, as well as (focusing on IP-based traffic) VoIP and WEB connections, can benefit from CAC. CAC implementation is often based on the availability, at each communication link, of a *feasibility region* $\Omega_{FR}$ in the call space [1], where given QoS requirements in terms of packet loss/packet delay probability are statistically guaranteed for each connection. No QoS is possible without controlling the call admission to a network as network resources (bandwidth and buffer space) are finite and must be shared among the connections currently present in the network. The simplest possible way of admitting or rejecting a new call into one link consists in accepting it if and only if the call state after its potential admittance is still within $\Omega_{FR}$. However, this *policy* (called *complete sharing*) may lead to a poor use of the resources [2]. This often motivates the consideration of other admission policies. A commonly used class of admission policies is represented by the *coordinate-convex (c.c.)* policies [2], which restrict $\Omega_{FR}$ to suitable subsets $\Omega$.

In general, finding optimal c.c. policies for a CAC problem is a difficult combinatorial optimization problem, even when $\Omega_{FR}$ is defined in terms of linear constraints (*stochastic knapsack* model). This case is investigated in [2, Chapter 4] and [3], where structural properties of the optimal c.c. policies are derived. When some form of *statistical multiplexing* [2, pp. 30-33] is used, the optimization problem is even more difficult, since in this case, typically, $\Omega_{FR}$ is characterized by nonlinear constraints [1]. Up to our knowledge, the problem of finding structural properties of the optimal c.c. policies has received little attention in the literature for the nonlinear case. One exception is [4], which investigates sufficient conditions

for the optimality of the complete-sharing policy in the context of wireless networks.

The letter is organized as follows (preliminary results appeared in a short abstract in [5]). In Section II we summarize the CAC model studied in [3], whereas in Section III we present our extension of this model to nonlinearly-constrained feasibility regions. In Subsections III-A, III-B, III-C, and III-D we provide, respectively:

- two criteria to improve certain suboptimal c.c. policies, based on the removal or addition of rectangular subregions near suitably-defined corner points;
- a characterization of the corner points for which the two above-mentioned criteria cannot be applied;
- a structural property holding for any c.c. policy that cannot be improved by exploiting the two criteria (in particular, valid in the case of an optimal c.c. policy);
- simulation results for the two criteria.

## II. BASIC PROBLEM FORMULATION

Following [3], the state of the CAC system is described by a 2-dimensional vector **n**, whose component $n_k$, $k = 1, 2$, represents the number of connections from users of class $k$, accepted and currently in progress. For each class $k$, inter-arrival times are exponentially distributed with mean values $1/\lambda_k(n_k)$. Holding times of the accepted connections are independent and identically distributed (i.i.d.) with mean $1/\mu_k$. The CAC system accepts or rejects a connection request according to a *c.c. policy*. We recall its definition [2].

**Definition II.1.** *A nonempty set $\Omega \subseteq \Omega_{FR} \subset \mathbb{N}_0^2$ is called c.c. iff it has the following property: for each $\mathbf{n} \in \Omega$ with $n_k > 0$ one has $\mathbf{n} - \mathbf{e}_k \in \Omega$, where $\mathbf{e}_k$ is a 2-dimensional vector whose $k$-th component is 1 and the other one is 0. A c.c. policy with associated c.c. set $\Omega$ admits an arriving request of connection iff the state process remains in $\Omega$ after admittance.*

As c.c. policies are in a one-to-one correspondence with c.c. sets, we use the symbol $\Omega$ for both. The objective to be maximized by the CAC system in the space $\mathcal{P}(\Omega_{FR})$ of c.c. subsets of $\Omega_{FR}$ is

$$J(\Omega) = \sum_{\mathbf{n} \in \Omega} (\mathbf{n} \cdot \mathbf{r}) P_{\Omega}(\mathbf{n}), \qquad (1)$$

where **r** is a 2-dimensional vector whose component $r_k$, $k = 1, 2$, represents the instantaneous positive revenue generated by any accepted connection of class $k$ that is still in progress and $P_{\Omega}(\mathbf{n})$ is the steady-state probability that the CAC system is in state **n** under the c.c. policy $\Omega$.

For linearly-constrained feasibility regions $\Omega_{FR}$, the previously-described model is called *stochastic knapsack*. For

such a case, in [3], structural properties of the c.c. policies maximizing the objective (1) (*optimal* c.c. policies) are investigated, e.g., the existence of vertical or horizontal thresholds.

## III. EXTENSION TO NONLINEARLY-CONSTRAINED $\Omega_{FR}$

We extend the model of [3] by allowing the set $\Omega_{FR}$ (assumed to be c.c.) to have a nonlinear upper boundary, denoted by $(\partial \Omega_{FR})^+$ (see Fig. 1(a)). Similarly, we denote by $(\partial \Omega)^+$ the (linear or nonlinear) upper boundary of $\Omega$.

### A. Two criteria to improve suboptimal c.c. policies

We start our study by providing two criteria to establish if a c.c. policy is suboptimal and to obtain an improved one. A c.c. policy $\Omega_1$ *improves* a c.c. policy $\Omega_2$ iff $J(\Omega_1) > J(\Omega_2)$. We recall the following two definitions from [3].

**Definition III.1.** *The tuple* $(\alpha, \beta) \in \Omega_{FR} \setminus \Omega$ *is a type-1 corner point for* $\Omega$ *iff* $\beta \geq 1$, $(\alpha, \beta - 1) \in \Omega$, *and either* $\alpha = 0$ *or* $(\alpha - 1, \beta) \in \Omega$. *The tuple* $(\alpha, \beta) \in \Omega_{FR} \setminus \Omega$ *is a type-2 corner point for* $\Omega$ *iff* $\alpha \geq 1$, $(\alpha - 1, \beta) \in \Omega$, *and either* $\beta = 0$ *or* $(\alpha, \beta - 1) \in \Omega$.

**Definition III.2.** *A nonempty set* $S^- \subset \Omega_{FR}$ *is incrementally removable with respect to* $\Omega$ *($IR_\Omega$) iff* $S^- \subset \Omega$ *and* $\Omega \setminus S^-$ *is still a c.c. set. A nonempty set* $S^+ \subset \Omega_{FR}$ *is incrementally admissible with respect to* $\Omega$ *($IA_\Omega$) iff* $S^+ \cap \Omega = \emptyset$ *and* $\Omega \cup S^+$ *is still a c.c. set.*

Loosely speaking, the next Proposition III.3 states the following. Given a c.c. policy $\Omega$, if for some type-2 corner point one can find two rectangular regions $S^+$ and $S^-$ (of the below-defined shape) as in Fig. 1(b), then $\Omega$ is suboptimal, since it can be improved either by adding $S^+$ or by removing $S^-$. This is a nontrivial result, since for the objective (1) and any two c.c. sets $\Omega_1, \Omega_2 \subseteq \Omega_{FR}$, the relationship $\Omega_1 \subseteq \Omega_2$ does not imply $J(\Omega_1) \leq J(\Omega_2)$. Note that, although Proposition III.3 looks similar to [3, Lemma 2], it states a different concept. While [3, Lemma 2] states a property satisfied by any optimal c.c. policy, Proposition III.3 provides a constructive way to improve certain suboptimal c.c policies.

**Proposition III.3.** *Let* $(\alpha, \beta)$ *be a type-2 corner point for* $\Omega$ *and suppose that there exist* $n, m, p \in \mathbb{N}_0$ *such that* $S^- := \{(\alpha - 1 - j, \beta + i) : j = 0, \ldots, n, i = 0, \ldots, p\} \subset \Omega$, *is* $IR_\Omega$, *and* $S^+ := \{(\alpha + s, \beta + i) : s = 0, \ldots, m, i = 0, \ldots, p\} \subset \Omega_{FR}$, *is* $IA_\Omega$. *Then, at least one of the following inequalities holds: (i)* $J(\Omega \cup S^+) > J(\Omega)$; *(ii)* $J(\Omega \setminus S^-) > J(\Omega)$.

**Proof.** Recall that, for a not necessarily c.c. set $S \subseteq \Omega_{FR}$, one can define $J(S) := H(S)/G(S)$, with $H(S)$ and $G(S)$ expressed in terms of suitable summations[1] over the elements of $S$. The definition coincides with (1) for a c.c. set. If $S := \{a, a+1, \ldots, b\} \times \{c, c+1, \ldots, d\}$, then

$$J(S) = r_1 x_1(a, b) + r_2 x_2(c, d). \qquad (2)$$

Now, suppose that neither $J(\Omega \cup S^+) > J(\Omega)$ nor $J(\Omega \setminus S^-) > J(\Omega)$ holds. Then

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[1]Due to the page limits, we refer to [3] for the expressions of the functions $H(\cdot)$ and $G(\cdot)$, and of $P_\Omega(\cdot)$ and $x_i(\cdot, \cdot)$ in formulas (1) and (2).
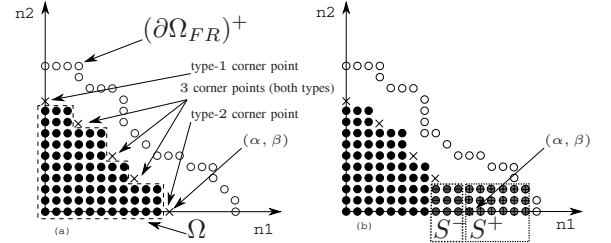


Fig. 1.   The sets $\Omega$, $S^+$ and $S^-$ in Proposition III.3.

$$J(\Omega \cup S^+) = \frac{H(\Omega) + H(S^+)}{G(\Omega) + G(S^+)} \leq J(\Omega) = \frac{H(\Omega)}{G(\Omega)},$$

which implies $J(S^+) = H(S^+)/G(S^+) \leq H(\Omega)/G(\Omega) = J(\Omega)$. Similarly, one gets $J(S^-) \geq J(\Omega)$, so $J(S^-) \geq J(S^+)$. On the other hand, computing $J(S^-)$ and $J(S^+)$ by formula (2) one has $J(S^-) = r_1 x_1(\alpha - 1 - n, \alpha - 1) + r_2 x_2(\beta, \beta + p)$, $J(S^+) = r_1 x_1(\alpha, \alpha + m) + r_2 x_2(\beta, \beta + p)$, thus $J(S^-) < J(S^+)$, but this is a contradiction. So, at least one between cases (i) and (ii) holds.                                    ∎

Proposition III.4 states a similar concept for type-1 corner points and is proved by reversing the roles of the two classes.

**Proposition III.4.** *Let* $(\alpha, \beta)$ *be a type-1 corner point for* $\Omega$ *and suppose that there exist* $n, m, p \in \mathbb{N}_0$ *such that* $S^- := \{(\alpha + i, \beta - 1 - j) : i = 0, \ldots, p, j = 0, \ldots, n\} \subset \Omega$, *is* $IA_\Omega$, *and* $S^+ := \{(\alpha + i, \beta + s) : i = 0, \ldots, p, s = 0, \ldots, m\} \subset \Omega_{FR}$, *is* $IA_\Omega$. *Then, at least one of the following inequalities holds: (i)* $J(\Omega \setminus S^-) > J(\Omega)$; *(ii)* $J(\Omega \cup S^+) > J(\Omega)$.

### B. A property of the corner points of any optimal c.c. policy

Let $\Omega^*$ denote any c.c. policy that cannot be further improved via Proposition III.3 or III.4. Our next Proposition III.5 characterizes the corner points for which neither Proposition III.3 nor III.4 can be applied (see Fig. 2(a) for an example). This is an interesting result since any such c.c. policy $\Omega^*$ and any *optimal* c.c. policy (which, obviously, cannot be further improved either with Proposition III.3 and III.4 or with other method), can have only this kind of corner points. We let

$$l_2^\Omega(n_1) := \max\{k \in \mathbb{N}_0 \text{ such that } (n_1, k) \in \Omega\}, \qquad (3)$$
$$l_1^\Omega(n_2) := \max\{h \in \mathbb{N}_0 \text{ such that } (h, n_2) \in \Omega\}. \qquad (4)$$

The values $l_1^\Omega(n_2)$ and $l_2^\Omega(n_1)$ are the maximum numbers of type-1 and type-2 connections allowed in $\Omega$ when one already has $n_2$ type-2 and $n_1$ type-1 connections, respectively. By their definitions, the functions $l_i^\Omega(\cdot)$ are non-increasing.

**Proposition III.5.** *The following hold.*

*(i) Let* $(\alpha, \beta)$ *be a type-2 corner point of* $\Omega$ *for which Proposition III.3 cannot be applied. Then* $l_2^\Omega(\alpha - 1) > l_2^{\Omega_{FR}}(\alpha)$.

*(ii) Let* $(\alpha, \beta)$ *be a type-1 corner point of* $\Omega$ *for which Proposition III.4 cannot be applied. Then* $l_1^\Omega(\beta - 1) > l_1^{\Omega_{FR}}(\beta)$.

**Proof.** We prove (i); for (ii), similar arguments can be used. The sets $S^+$ and $S^-$ in Proposition III.3 are rectangles with the same height $p$. The only value of $p$ for which $S^-$ is $IR_\Omega$ is $p = l_2^\Omega(\alpha - 1)$. The maximum possible value of $p$ for which
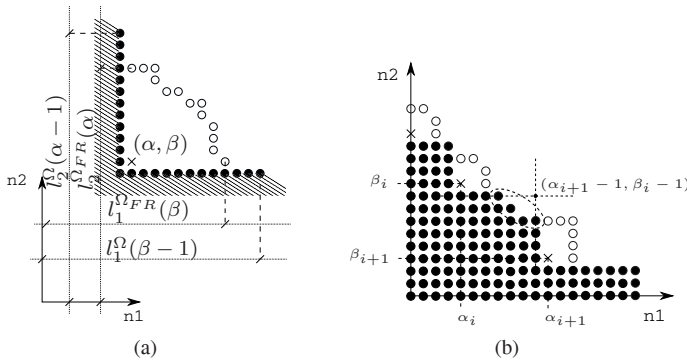
Fig. 2. (a) An example of a corner point $(\alpha, \beta)$ for which neither Proposition III.3 nor III.4 can be applied. (b) An example of a c.c. policy $\Omega^*$ that cannot be improved by applying Propositions III.3 and III.4.

$S^+$ is $IA_\Omega$ is $p = l_2^{\Omega_{FR}}(\alpha)$. So, if $l_2^\Omega(\alpha-1) > l_2^{\Omega_{FR}}(\alpha)$, then Proposition III.3 cannot be applied. If, instead, $l_2^\Omega(\alpha-1) \leq l_2^{\Omega_{FR}}(\alpha)$, then one can find sets $S^-$ and $S^+$ that satisfy its assumptions (e.g., $p = l_2^\Omega(\alpha-1)$, $n = m = 0$). ∎

### C. A structural property of any optimal c.c. policy

The next Theorem III.6, which is our main result, provides a characterization of the upper boundary $(\partial\Omega^*)^+$ of $\Omega^*$ and its intersection with $(\partial\Omega_{FR})^+$, when $\Omega^*$ has at least two corner points. Theorem III.6 implies that between any two successive corner points the intersection between $(\partial\Omega^*)^+$ and $(\partial\Omega_{FR})^+$ is nonempty (see the dotted ellipse in Fig. 2(b)). Up to our knowledge, no such result was previously available in the literature, even for linearly-constrained feasibility regions.

**Theorem III.6.** *Let $(\alpha_i, \beta_i)$ and $(\alpha_{i+1}, \beta_{i+1})$ be two consecutive corner points of $\Omega^*$. Then the intersection between the vertical line $n_1 = \alpha_{i+1} - 1$ and the horizontal line $n_2 = \beta_i - 1$ either lies on $(\partial\Omega_{FR})^+$ or is outside $\Omega_{FR}$.*

**Proof.** The claim is equivalent to the pair of inequalities

$$l_1^{\Omega_{FR}}(\beta_i - 1) \leq \alpha_{i+1} - 1, \tag{5}$$

$$l_2^{\Omega_{FR}}(\alpha_{i+1} - 1) \leq \beta_i - 1. \tag{6}$$

Let us prove (5). By the definition of $l_2^{\Omega^*}(\alpha_i)$, the monotonicity of $l_2^{\Omega^*}(\cdot)$, and Proposition III.5 (i), we get

$$\beta_i - 1 = l_2^{\Omega^*}(\alpha_i) \geq l_2^{\Omega^*}(\alpha_{i+1} - 1) > l_2^{\Omega_{FR}}(\alpha_{i+1}). \tag{7}$$

Now, suppose that the inequality $l_1^{\Omega_{FR}}(\beta_i - 1) > \alpha_{i+1} - 1$, opposite to (5), holds. Let us show that this leads to a contradiction. As $\alpha_{i+1}$ is an integer, one has

$$l_1^{\Omega_{FR}}(\beta_i - 1) > \alpha_{i+1} - 1 \Leftrightarrow l_1^{\Omega_{FR}}(\beta_i - 1) \geq \alpha_{i+1}.$$

This, combined with the property $l_2^{\Omega_{FR}}(l_1^{\Omega_{FR}}(\beta_i-1)) \geq \beta_i - 1$ (which is a consequence of (3) and (4)) and the monotonicity of $l_2^{\Omega_{FR}}(\cdot)$, implies $l_2^{\Omega_{FR}}(\alpha_{i+1}) \geq \beta_i - 1$, but this contradicts (7). So, (5) must hold. The proof of (6) is similar. ∎

### D. Simulation results

To verify the effectiveness of the criteria proposed in Subsection III-A, we provide some simulation results. We consider a cell in a cellular network and two classes of traffic, i.e.,
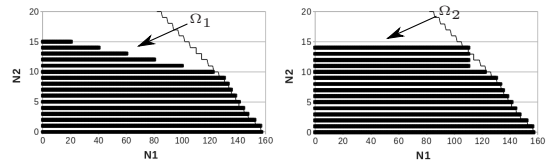


Fig. 3. (a) initial c.c. policy $\Omega_1$; (b) final c.c. policy $\Omega_2$.

voice call traffic (class 1) and data traffic (class 2), modelled by Poisson arrivals and exponential call durations (as often done in the literature; see, e.g., [6]). For the class-1 traffic we have (on average) 20 calls per time unit, e.g., per minute ($\lambda_1 = 20$), with an average holding time of 3 time units per call ($\mu_1 = 1/3$). For class-2 traffic we set $\lambda_2 = 10$ and $\mu_2 = 1/20$. The per-call instantaneous revenues of the two classes are the same ($r_1 = r_2 = 1$). The $\Omega_{FR}$ used in the simulations has a nonlinear upper boundary that models QoS constraints, like the ones in [4, Fig. 3] and [1, pp. 46-49].

Starting from the initial c.c. policy $\Omega_1$ depicted in Fig. 3(a), the final c.c. policy $\Omega_2$ in Fig. 3(b) has been obtained by applying Proposition III.3 four times, to suitable corner points. The initial value of the objective is $J(\Omega_1) = 66.4229$, whereas the final value is $J(\Omega_2) = 73.9256$, with an improvement of 11.3%. Note from Fig. 3(b) that the c.c. policy $\Omega_2$ cannot be further improved via Propositions III.3 or III.4 and that its corner points have the structural properties stated in Proposition III.5 and Theorem III.6.

## IV. CONCLUSIONS

CAC is an important tool to guarantee QoS in telecommunications networks. The stochastic knapsack model, as well as our extension in Section III, can be used, e.g., to model CAC in telephone exchanges or in cellular networks. We have also provided a characterization of the optimal c.c. policies in CAC problems with nonlinearly-constrained feasibility regions. The results obtained in Subsections III-A, III-B, III-C can be applied to narrow the search for the (unknown) optimal c.c. policies and to improve certain suboptimal c.c. policies (see Subsection III-D). The two criteria proposed in Subsection III-A may also be integrated in local search or greedy algorithms and applied to problems with more than two classes of users (e.g., by defining subproblems obtained by partitioning the set of classes into subsets of cardinalities at most two).

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